

A non-conforming discontinuous Galerkin method for solving Maxwell's equations

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I - Discontinuous Galerkin method

The physical problem

The Maxwell equations in $\Omega \in \mathbb{R}^3$:

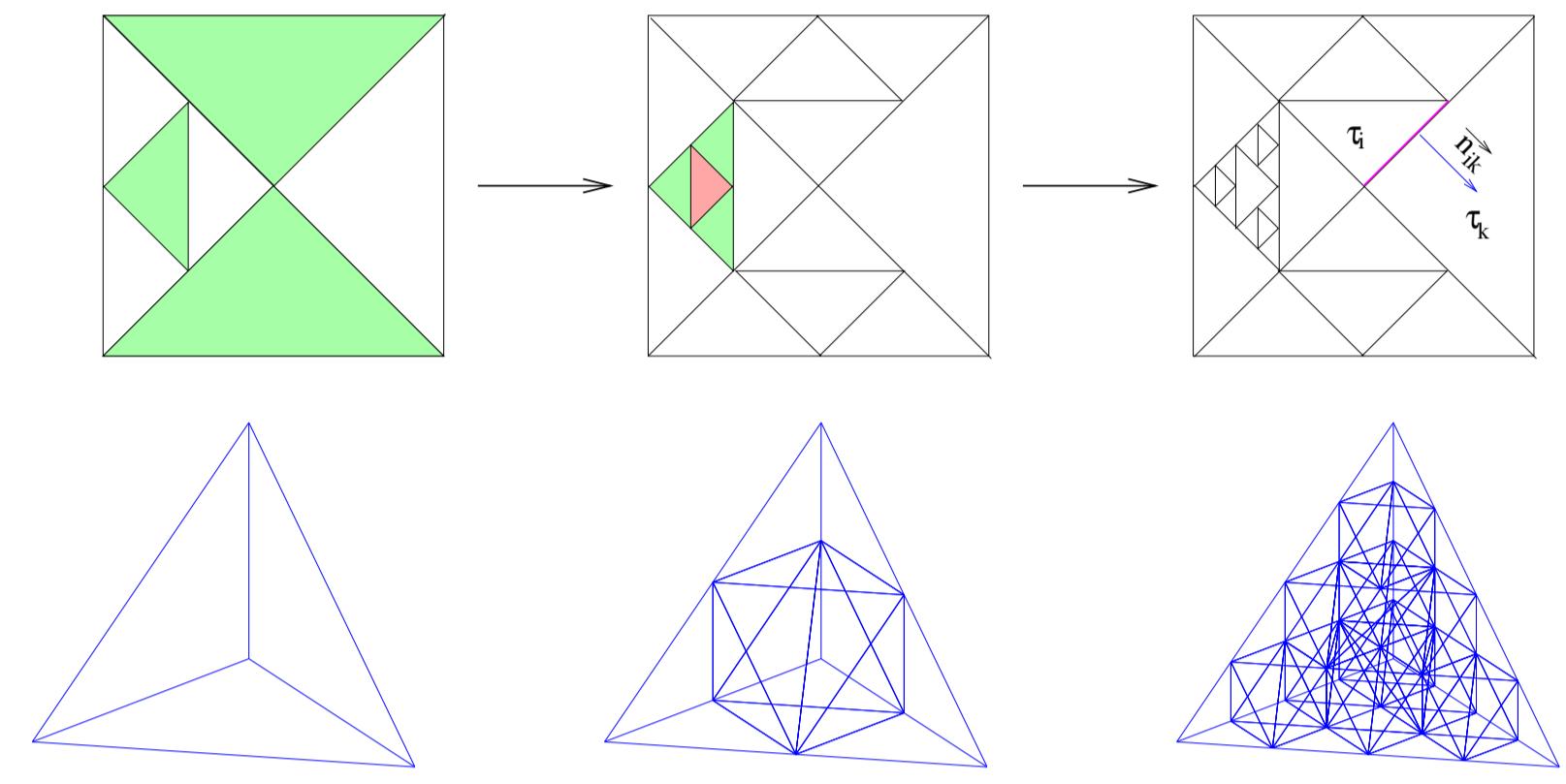
$$\begin{cases} \bar{\epsilon} \partial_t \vec{E} = \operatorname{curl} \vec{H}, \\ \bar{\mu} \partial_t \vec{H} = -\operatorname{curl} \vec{E}, \end{cases} \quad \text{and} \quad \begin{cases} \nabla \cdot \vec{E} = 0, \\ \nabla \cdot \vec{H} = 0, \end{cases}$$

• $\bar{\epsilon}(\vec{x})$ and $\bar{\mu}(\vec{x})$ are symmetric positive definite

• Boundary conditions : $\partial\Omega = \Gamma_a \cup \Gamma_m$

$$\begin{cases} \vec{n} \times \vec{E} = 0 \text{ on } \Gamma_m \\ \vec{n} \times \vec{E} = -c\mu \vec{n} \times (\vec{n} \times \vec{H}) \text{ on } \Gamma_a \end{cases}$$

Non-conforming simplicial meshes



• \mathcal{V}_i = set of indices of elements neighboring τ_i

• $a_{ik} = \tau_i \cap \tau_k$ (the interface)

• p_i is the polynomial degree defined inside τ_i

• $p = \{p_i : \tau_i \in \Omega_h\}$

• h_i is the size of τ_i and $h = \max_{\tau_i \in \mathcal{T}_h} h_i$

• $V_p(\Omega_h) := \{\vec{v} \in L^2(\Omega)^3 : \vec{v}|_{\tau_i} \in \mathbb{P}_{p_i}(\tau_i), \forall \tau_i \in \Omega_h\}$

Space discretization

• $\forall \vec{\varphi} \in \operatorname{Span}\{\varphi_{ij}, 1 \leq j \leq d_i\}$

$$\begin{cases} \int_{\tau_i} \vec{\varphi} \cdot \bar{\epsilon}_i \partial_t \vec{E} = \int_{\tau_i} \operatorname{curl} \vec{\varphi} \cdot \vec{H} - \int_{\partial \tau_i} \vec{\varphi} \cdot (\vec{H} \times \mathbf{n}) \\ \int_{\tau_i} \vec{\varphi} \cdot \bar{\mu}_i \partial_t \vec{H} = -\int_{\tau_i} \operatorname{curl} \vec{\varphi} \cdot \vec{E} + \int_{\partial \tau_i} \vec{\varphi} \cdot (\vec{E} \times \mathbf{n}) \end{cases}$$

• Centered numerical fluxes

$$\vec{E}|_{a_{ik}} = \frac{\vec{E}_i + \vec{E}_k}{2}, \quad \vec{H}|_{a_{ik}} = \frac{\vec{H}_i + \vec{H}_k}{2}$$

• Local matrix form of the DGTD- \mathbb{P}_p scheme :

$$\begin{cases} M_i^\epsilon \partial_t \mathbf{E}_i = K_i \mathbf{H}_i - \sum_{k \in \mathcal{V}_i} S_{ik} \mathbf{H}_k \\ M_i^\mu \partial_t \mathbf{H}_i = -K_i \mathbf{E}_i + \sum_{k \in \mathcal{V}_i} S_{ik} \mathbf{E}_k \end{cases}$$

– M_i^ϵ & M_i^μ are the mass matrices of size d_i

– K_i is the stiffness matrix of size d_i

– S_{ik} is the interface matrix of size $d_i \times d_k$

$$(S_{ik})_{jl} = \frac{1}{2} \int_{a_{ik}} \vec{\varphi}_{ij} \cdot (\vec{\varphi}_{kl} \times \vec{n}_{ik})$$

* For only non-conforming interface $a_{ik} \Rightarrow S_{ik}$ is evaluated using cubature formulas

• 2D : Gauss-Legendre quadrature

• 3D : Dunavant cubature formula

Time discretization

High-order leap-frog (LF_N) time scheme

• General form of the DGTD- \mathbb{P}_p method :

$$\begin{cases} \frac{M^\epsilon E^{n+1} - E^n}{\Delta t} = S_N H^{n+\frac{1}{2}} \\ \frac{M^\mu H^{n+\frac{3}{2}} - H^{n+\frac{1}{2}}}{\Delta t} = -t S_N E^{n+1} \end{cases}$$

where the $d \times d$ matrix S_N verifies:

$$S_N = \begin{cases} S & \text{if } N = 2 \\ S(I - \frac{\Delta t^2}{24} M^{-\mu} t S M^{-\epsilon} S) & \text{if } N = 4 \end{cases}$$

– E and H are of size $d = \sum_i d_i$

– M^ϵ and M^μ are block diagonal mass matrices of size d with diagonal blocks equal to M_i^ϵ and M_i^μ respectively

II - Stability & convergence analysis

Stability

The EM energy: $\mathcal{E}^n = \frac{1}{2} (\mathbb{E}^n M^\epsilon \mathbb{E}^n + t \mathbb{H}^{n-\frac{1}{2}} M^\mu \mathbb{H}^{n+\frac{1}{2}})$

• The energy \mathcal{E}^n is stable if:

$$\Delta t \leq \frac{2}{d_N}, \quad \text{with} \quad d_N = \|M^{\frac{-\mu}{2}} t S_N M^{\frac{-\epsilon}{2}}\|$$

• $\operatorname{CFL}(\operatorname{LF}_N) = \nu_N \operatorname{CFL}(\operatorname{LF}_2)$

N	2	4	6	8	10	12	14	16	18	20
ν_N	1	2.85	3.68	3.79	5.27	4.44	6.42	7.53	7.27	8.91

Convergence

• Exact solution : $\vec{U}(t) = (\vec{E}(t), \vec{H}(t)) \in [\mathbb{H}^s(\tau_i)]^2$

• Numer. solution : $\vec{U}_h(t) = (\vec{E}_h(t), \vec{H}_h(t)) \in [V_p(\Omega_h)]^2$

$$\|\vec{U} - \vec{U}_h\|_{0,\Omega} \leq C \frac{h^\nu}{p^s} \left(1 + T \frac{p^{\frac{5}{2}}}{h}\right) \max_{t \in [0,T]} \|\vec{U}(t)\|_{s,\Omega}$$

Convergence of the divergence error

$$\|\nabla \cdot (\vec{U} - \vec{U}_h)\|_{0,\Omega} \leq C \frac{h^{\nu-1}}{p^{s-1}} \left(1 + T \frac{p^{\frac{5}{2}}}{h}\right) \max_{t \in [0,T]} \|\vec{U}(t)\|_{s,\Omega}$$

• $\nu = \min\{s, p+1\}$

• $C(\epsilon, \mu)$ independent of h and p

III - Numerical experiments

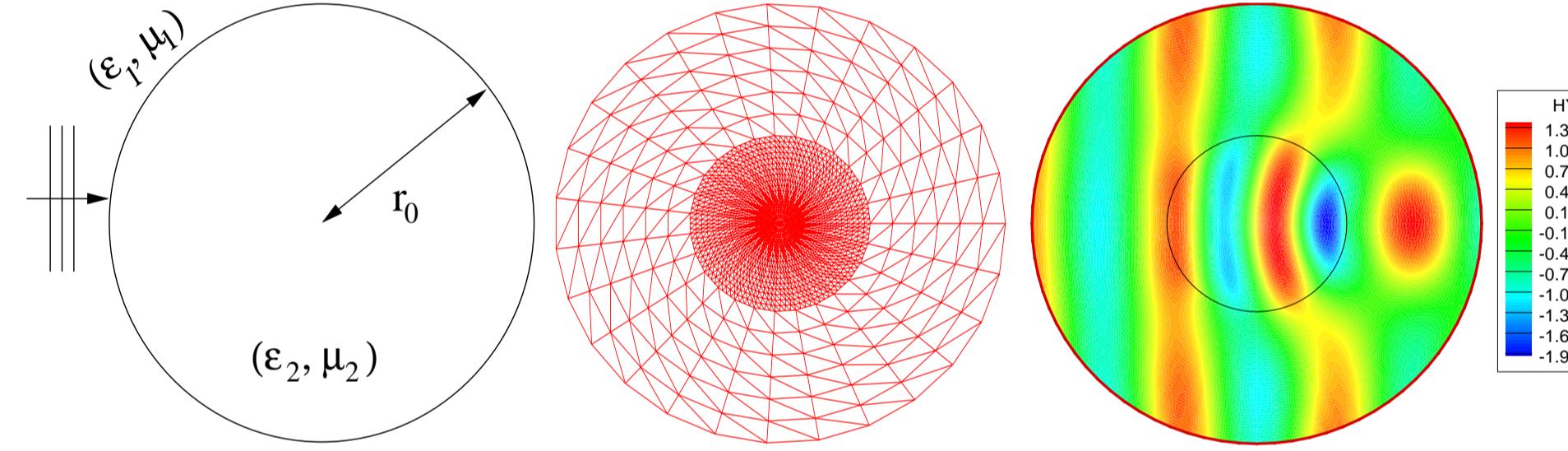
Scattering by a dielectric cylinder

• $\mu_1 = \mu_2 = 1$

• $r_0 = 0.6 \text{ m}$, freq = 300 MHz

• $\epsilon_1 = 1, \epsilon_2 = 2.25$

• $\Omega = \{r \leq 1.5 \text{ m}\}$



DGTD- \mathbb{P}_p METHOD : CONFORMING MESH 11920 TRIANGLES & 6001 NODES				
method	DGTD- \mathbb{P}_0	DGTD- \mathbb{P}_1	DGTD- \mathbb{P}_2	DGTD- \mathbb{P}_3
L^2 error, CPU (min)	13.6%, 20	7.15%, 178	5.20%, 542	5.22%, 1817
# DOF	11920	35760	71520	119200

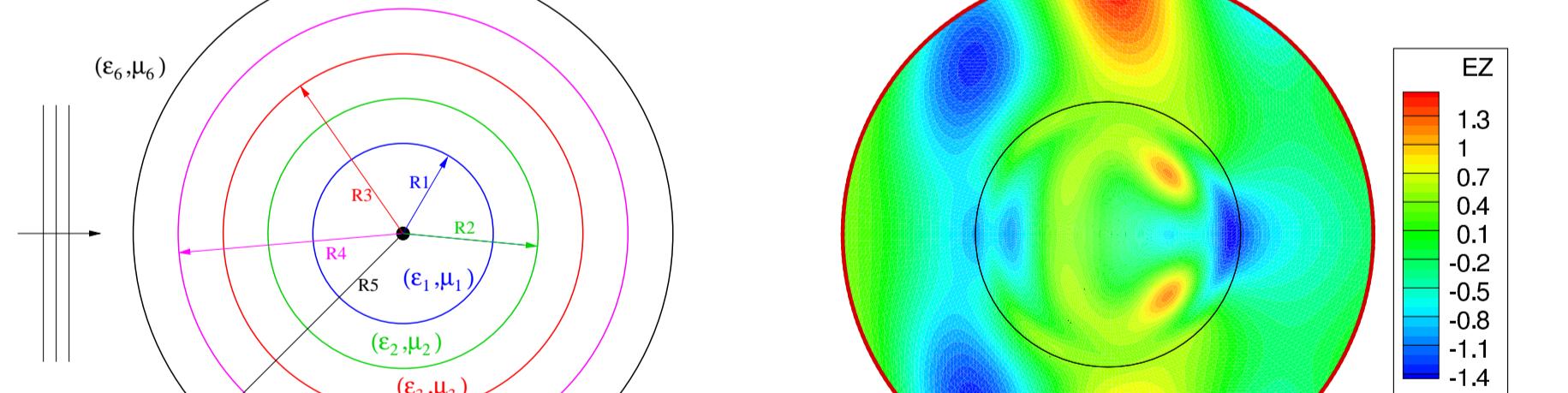
DGTD- $\mathbb{P}_{(p_1,p_2)}$ METHOD : NON-CONFORMING MESH 5950 TRIANGLES & 3151 NODES (300 HANGING NODES)				
method	DGTD- $\mathbb{P}_{(1,0)}$	DGTD- $\mathbb{P}_{(2,0)}$	DGTD- $\mathbb{P}_{(2,1)}$	DGTD- $\mathbb{P}_{(3,2)}$
L^2 error, CPU (min)	11.6%, 9	5.36%, 25	5.39%, 33	5.37%, 179
# DOF	11450	19700	26100	46700

Scattering by multilayered dielectric cylinders

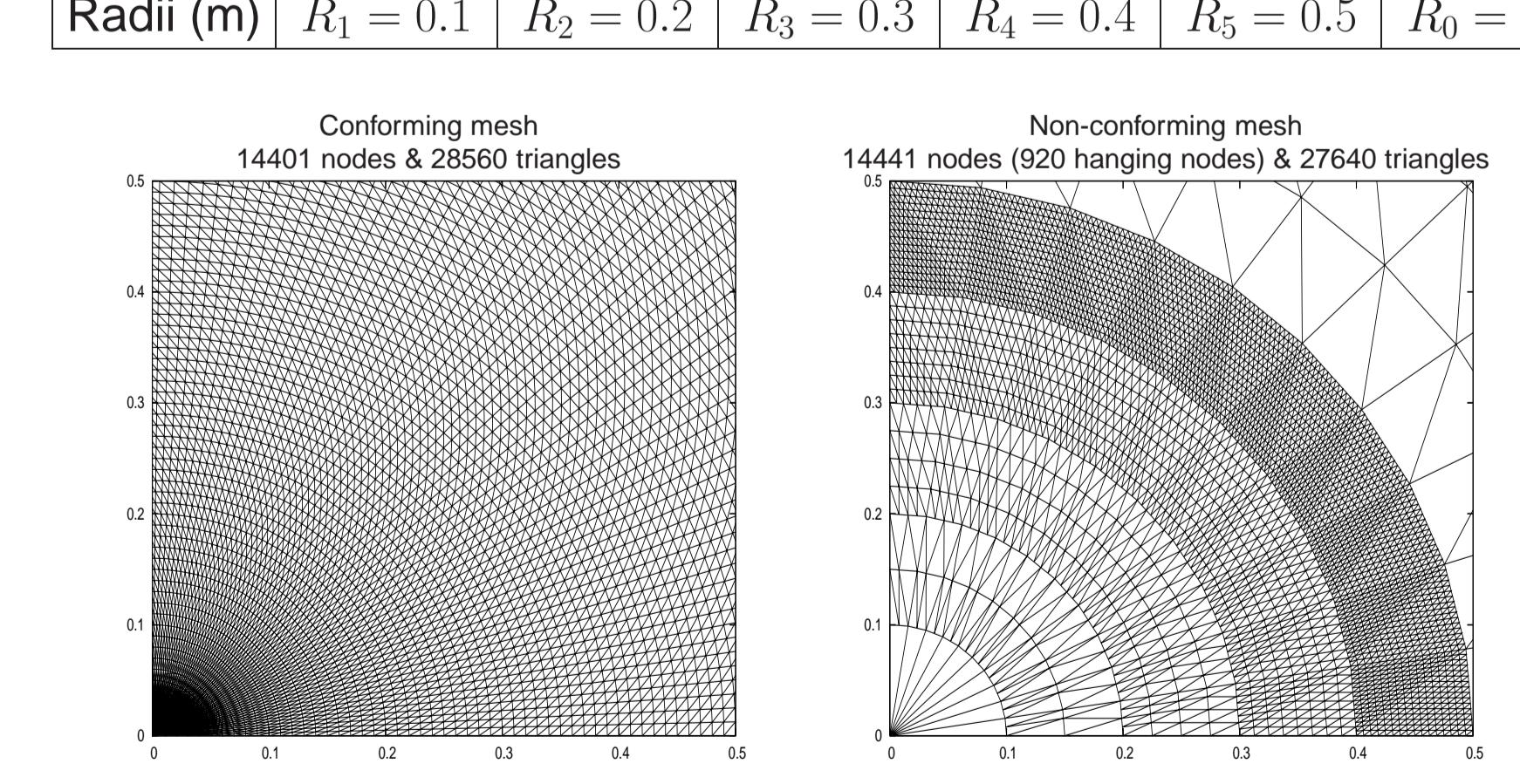
The cylinder is illuminated by a monochromatic plane wave of the form:

$$E_z^{inc} = \exp(-i(k_0 x - \omega t)) \quad \& \quad H_y^{inc} = -\exp(-i(k_0 x - \omega t))$$

$$k_0 = \omega / \sqrt{\epsilon_0 \mu_0} \quad \& \quad \text{freq} = 300 \text{ MHz}$$



Region	Region 1	Region 2	Region 3	Region 4	Region 5	Region 6
ϵ_r	$\epsilon_1 = 1$	$\epsilon_2 = 4$	$\epsilon_3 = 9$	$\epsilon_4 = 16$	$\epsilon_5 = 64$	$\epsilon_6 = 1$



$p_i = p$	Error on H_y	Error on E_z	CPU time	# DOF
1	7.6 %	7.80 %	137 min	85680
2	2.2 %	1.30 %	286 min	171360
3	2.2 %	1.20 %	842 min	285600

$p_1, p_2, p_3, p_4, p_5, p_6$	Error on H_y	Error on E_z	CPU time	# DOF
4,3,2,1,0,2	5.0 %	1.7 %	12 min	49720
4,3,2,2,1,4				