A discontinuous Galerkin method for solving the 2D time-domain Maxwell's equations on non-conforming locally refi ned meshes

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I - General introduction

Summary

This work is concerned with the design of a hp-like discontinuous Galerkin (DG) method for solving the 2D time-domain Maxwell's equations on non-conforming locally refined triangular meshes. The proposed DG method allows non-conforming meshes with arbitrary-level hanging nodes. This method combines a centered approximation for the evaluation of fluxes at the interface between neighboring elements of the mesh, with a leap-frog time integration scheme.

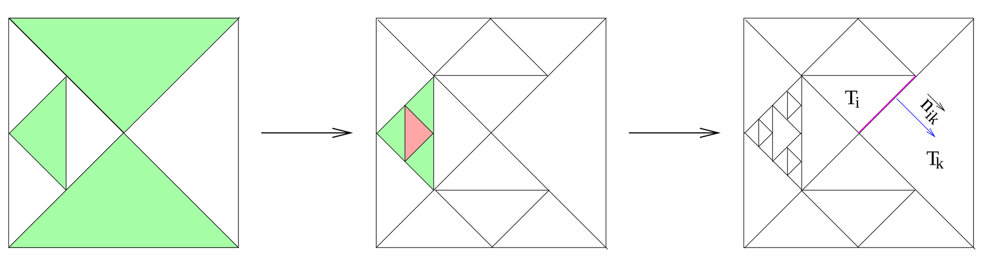
Maxwell's equations

The 2D Maxwell's equation in the TM polarization:

$$\begin{cases} \epsilon \dfrac{\partial E_z}{\partial t} - \dfrac{\partial H_y}{\partial x} + \dfrac{\partial H_x}{\partial y} = 0, \\ \mu \dfrac{\partial H_x}{\partial t} + \dfrac{\partial E_z}{\partial y} = 0, \\ \mu \dfrac{\partial H_y}{\partial t} - \dfrac{\partial E_z}{\partial x} = 0, \end{cases}$$

- E_z and $\mathbf{H} = (H_x, H_y)$ are respectively the electric and magnetic fields
- ϵ and μ are respectively the electric permittivity and magnetic permeability of the medium; they are assumed to be piecewise constant

Non-conforming locally refined meshes



Non-conforming meshes where triangle vertices can lie in the interior of edges of other triangles

- $\mathcal{V}_i$ = set of indices of elements neighboring T_i
- $s_{ik} = T_i \bigcap T_k$ (the interface)
- p_i is an integer assigned to the element T_i
- h_i is the size of T_i and $h = \max_{T_i \in \mathcal{T}_h} h_i$

Boundary conditions

- PEC (metallic) boundary conditions: $E_z = 0$
- First order Silver-Müller (absorbing) conditions:

$$E_z = c\mu(n_y H_x - n_x H_y)$$

Numerical flux and time-integration scheme

- Totally centered numerical fluxes at the interface between elements
- Second order leap-frog time-scheme, i.e. E_z is computed at integer time-stations and H_x and H_y at half-integer time-stations

The DGTD scheme

The DGTD-$\mathbb{P}_{p_i}$ method writes:

$$\begin{aligned} \epsilon_i \mathbb{M}_i \frac{\mathbf{E}_{z_i}^{n+1} - \mathbf{E}_{z_i}^{n}}{\Delta t} &= -\mathbb{K}_i^x \mathbf{H}_{y_i}^{n+\frac{1}{2}} + \mathbb{K}_i^y \mathbf{H}_{x_i}^{n+\frac{1}{2}} \\ &\quad + \sum_{k \in \mathcal{V}_i} \left(\mathbb{G}_{x_{ik}}^{n+\frac{1}{2}} - \mathbb{G}_{y_{ik}}^{n+\frac{1}{2}} \right), \\ \mu_i \mathbb{M}_i \frac{\mathbf{H}_{x_i}^{n+\frac{1}{2}} - \mathbf{H}_{x_i}^{n-\frac{1}{2}}}{\Delta t} &= \mathbb{K}_i^y \mathbf{E}_{z_i}^{n} - \sum_{k \in \mathcal{V}_i} \mathbb{F}_{y_{ik}}^{n}, \\ \mu_i \mathbb{M}_i \frac{\mathbf{H}_{y_i}^{n+\frac{1}{2}} - \mathbf{H}_{y_i}^{n-\frac{1}{2}}}{\Delta t} &= -\mathbb{K}_i^x \mathbf{E}_{z_i}^{n} + \sum_{k \in \mathcal{V}_i} \mathbb{F}_{x_{ik}}^{n}, \end{aligned}$$

- $\mathbb{M}_i$ is the constant local mass (symmetric positive definite) matrix, and $\mathbb{K}_i^x$ is the constant (skew-symmetric) stiffness matrix
- The vector $\mathbb{F}_{\mathbf{x}_{ik}}^{n}$ and $\mathbb{G}_{\mathbf{x}_{ik}}^{n+\frac{1}{2}}$ are defined as:

$$\mathbb{F}_{\mathbf{x}_{ik}}^{n} = \mathbb{S}_{ik}^{\mathbf{x}} \mathbf{E}_{z_k}^{n} \text{ and } \mathbb{G}_{\mathbf{x}_{ik}}^{n+\frac{1}{2}} = \mathbb{S}_{ik}^{\mathbf{x}} \mathbf{H}_{\mathbf{x}_k}^{n+\frac{1}{2}},$$

- $\mathbb{S}_{ik}^{\mathbf{x}}$ is the $d_i \times d_k$ interface matrix on s_{ik}
- For non-conforming interfaces, $\mathbb{S}_{ik}^{\mathbf{x}}$ is computed by using a Gaussian quadrature formula

The hp-like DGTD method

- Combines h-refinement with p-enrichment
- Consists in using high polynomial orders (p_c) in the coarse mesh and low order (p_f) in the fined one (e.g. the refined elements)
- The resulting scheme is called the DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method
- If $p_c = p_f = p$, the scheme is named the DGTD-$\mathbb{P}_p$

II - Stability analysis

On any non-conforming mesh, the DGTD method exactly conserves the following energy [1]:

$$2\mathcal{E}^n = \sum_i \epsilon_i {}^t\mathbf{E}_{z_i}^{n} \mathbb{M}_i \mathbf{E}_{z_i}^{n} + \mu_i {}^t\mathbf{H}_{x_i}^{n-\frac{1}{2}} \mathbb{M}_i \mathbf{H}_{x_i}^{n+\frac{1}{2}} + \mu_i {}^t\mathbf{H}_{y_i}^{n-\frac{1}{2}} \mathbb{M}_i \mathbf{H}_{y_i}^{n+\frac{1}{2}}$$

The energy $\mathcal{E}^n$ is a positive definite quadratic form of all numerical unknowns under the CFL-like sufficient stability condition on the time step Δt:

$$\forall i,\ \forall k \in \mathcal{V}_i,\quad c_i \Delta t (2\alpha_i + \beta_{ik}) \leq 4 \min\left(\frac{|T_i|}{P_i^x}, \frac{|T_i|}{P_i^y} \right),$$

where c_i is the local speed of propagation, $|T_i|$ is the surface of T_i and $P_i^{\mathbf{x}} = \sum_{k \in \mathcal{V}_i} |n_{ik\mathbf{x}}|$. The constants α_i and β_{ik} ($k \in \mathcal{V}_i$) verify:

$$\begin{cases} \forall \xi_i \in \text{Span}\{\varphi_{ij}, 1 \leq j \leq d_i\}\ ,\ \mathbf{x} \in \{x, y\} \\ \left\| \dfrac{\partial \xi_i}{\partial \mathbf{x}} \right\|_{T_i} \leq \dfrac{\alpha_i P_i^{\mathbf{x}}}{|T_i|} \|\xi_i\|_{T_i},\quad \|\xi_i\|_{s_{ik}}^2 \leq \beta_{ik} \dfrac{\|\vec{n}_{ik}\|}{|T_i|} \|\xi_i\|_{T_i}^2. \end{cases}$$

NUMERICAL CFL OF THE DGTD-$\mathbb{P}_p$ METHOD					
p_c=p_f=p	0	1	2	3	4
ν_p^{num}	1.0	0.3	0.2	0.15	0.1

NUMERICAL CFL OF THE DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ METHOD									
$p_c : p_f$	1:0	2:0	3:0	4:0	2:1	3:1	4:1	3:2	4:2
$\nu_{p_c:p_f}^{\text{num}}$	1.0	0.4	0.3	0.18	0.3	0.25	0.18	0.15	0.15

III - Convergence analysis

In [2] it is shown that the convergence order of the centered in space and time DGTD-$\mathbb{P}_p$ method, in the case of conforming simplicial meshes, is

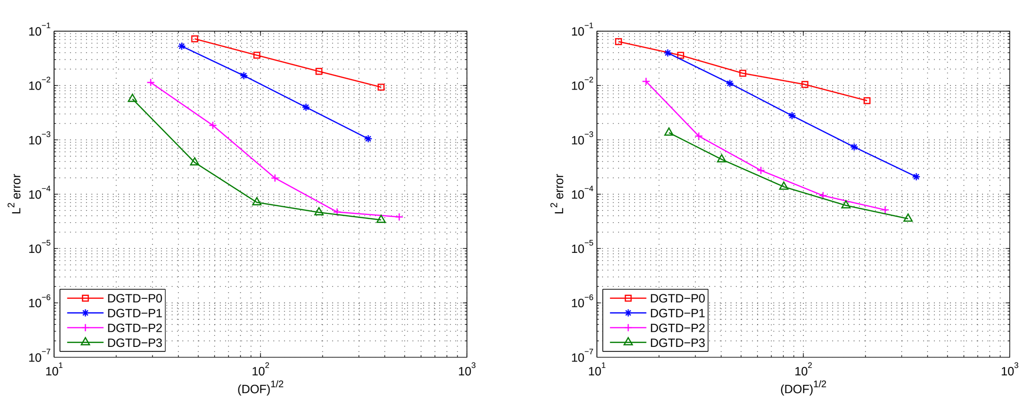
$$\mathcal{O}(Th^{\min(s,p)}) + \mathcal{O}(\Delta t^2),$$

Δt is the time step over the interval $[0, T]$ and $s > 1/2$ is such that the solution is sufficiently regular.

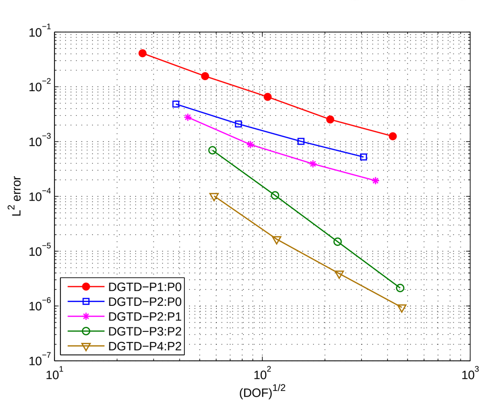
Resonance in a PEC cavity

- domain = $[0, 1]^2$
- (1,1)-eigenmode
- freq = 0.212 GHz
- $\epsilon = \mu = 1$
- refined zone = $[0.35, 0.65]^2$
- 7 hanging nodes per non-conforming interface
- errors and CPU times are measured at $T = 3$ ns

Numerical convergence of the DGTD-$\mathbb{P}_p$ method with h-refinement. Non-Conforming (left) and conforming (right) triangular mesh.



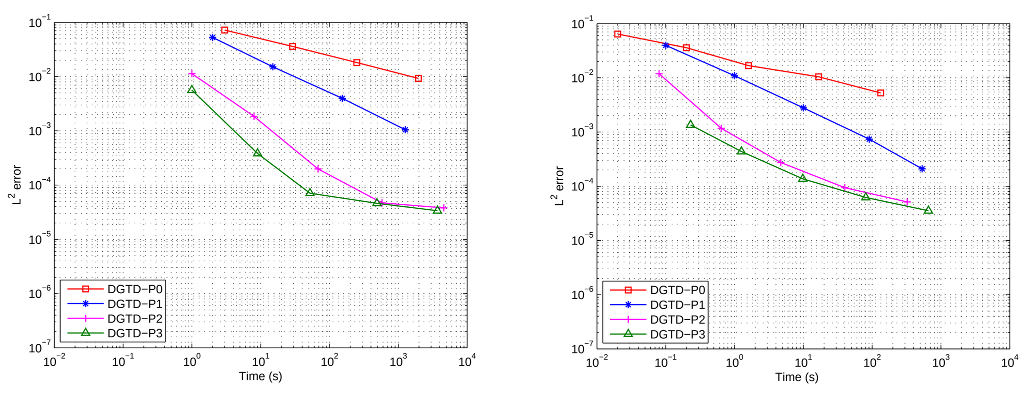
Numerical convergence of the DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method with hp-refined non-conforming triangular mesh



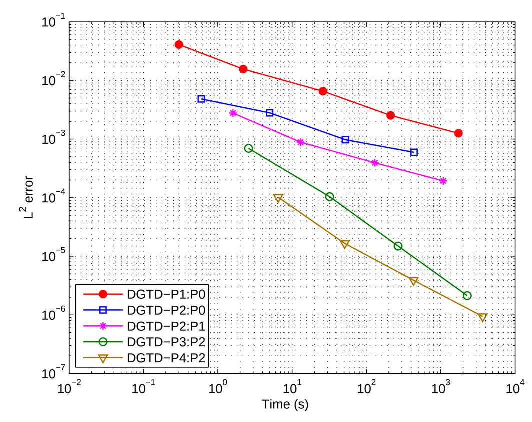
NUMERICAL CONVERGENCE ORDER OF THE DGTD-$\mathbb{P}_p$ METHOD								
	CONFORMING MESH				NON-CONFORMING MESH			
p	0	1	2	3	0	1	2	3
order	0.90	1.90	1.98	1.37	0.99	1.89	2.17	1.78

NUMERICAL CONVERGENCE ORDER OF THE DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ METHOD									
	NON-CONFORMING MESH								
p_c:p_f	1:0	2:0	3:0	4:0	2:1	3:1	4:1	3:2	4:2
order	1.27	1.06	0.98	1.07	1.27	1.01	1.0	2.78	2.24

L^2 error against CPU time using DGTD-$\mathbb{P}_p$ method with h-refinement. Non-Conforming (left) and conforming (right) triangular mesh.



L^2 error against CPU time using DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method with hp-refined non-conforming triangular mesh



Propagation of a Gaussian pulse

- $\Omega = [-2, 2] \times [0, 1]$
- $\Omega_1 = [-2, 0] \times [0, 1]$ = refined zone
- $\mu = 1$ in Ω, $\epsilon = 16$ in Ω_1, $\epsilon = 1$ in $\Omega \backslash \Omega_1$
- frequency = 1 GHz
- absorbant boundary conditions on $\partial\Omega$
- 3 hanging nodes per non-conforming interface
- errors and CPU times are measured at $T = 5$ ns

NUMERICAL CONVERGENCE ORDER OF THE DGTD-$\mathbb{P}_p$ METHOD								
	CONFORMING MESH				NON-CONFORMING MESH			
p	0	1	2	3	0	1	2	3
order	1.14	1.47	1.23	1.09	1.3	1.18	0.94	0.96

NUMERICAL CONVERGENCE ORDER OF THE DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ METHOD									
	NON-CONFORMING MESH								
p_c:p_f	1:0	2:0	3:0	4:0	2:1	3:1	4:1	3:2	4:2
order	1.3	1.56	1.64	1.76	1.44	1.44	1.57	0.93	0.94

CPU TIMES (MINUTES) AND NUMBER OF TIME STEPS (#STEP) OBTAINED TO ACHIEVE AN ACCURACY OF 5.0E-04 USING NON-CONFORMING MESH							
	DGTD-$\mathbb{P}_p$ METHOD			DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ METHOD			
p	1	2	3	p_c:p_f	2:0	3:2	4:1
CPU	24	21	15	CPU	18	19	17
#step	2723	2700	2459	#step	2439	2815	2699

Concluding remarks

A non-dissipative hp-like DGTD method is proposed for solving Maxwell's equations on both conforming and non-conforming locally refined triangular meshes. The unknowns are approximated with discontinuous nodal polynomials of degree that may vary over different elements of the mesh. This method conserves a discrete energy and is stable under a CFL condition. Numerical experiments prove the performance of the DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method and show that it can reduce the dispersion errors resulting from the classical DGTD-$\mathbb{P}_p$ $(p \leq 1)$ method.

References

[1] H. Fahs, S. Lanteri, and F. Rapetti, "A hp-like discontinuous Galerkin method for solving the 2D time-domain Maxwell's equations on non-conforming locally refined triangular meshes," **INRIA**, RR 6162, 2007.

[2] L. Fezoui, S. Lanteri, S. Lohrengel, and S. Piperno, "Convergence and stability of a discontinuous Galerkin time-domain method for the heterogeneous Maxwell equations on unstructured meshes," **M2AN**, vol. 39, no. 6, pp. 1149–1176, 2005.