

A high-order non-conforming discontinuous Galerkin method for time-domain electromagnetics

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Context

- Time-domain electromagnetic wave propagation
- Irregularly shaped geometries, heterogeneous media
 - Non-conforming, locally refined, triangular (2D)/tetrahedral (3D) meshes
- Numerical ingredients (starting point to this study)
[L. Fezoui, S. Lanteri, S. Lohrengel and S. Piperno: ESAIM, M2AN, 2005](#)
 - Discontinuous Galerkin time-domain (DGTD) methods
 - Nodal (Lagrange type) polynomial interpolation
 - Explicit time integration
- Overall objectives of this study
 - Investigate strengths and weaknesses of explicit DGTD methods using non-conforming simplicial meshes with arbitrary level hanging nodes
 - Theoretical and numerical aspects (stability, dispersion error, convergence)
 - Computational aspects

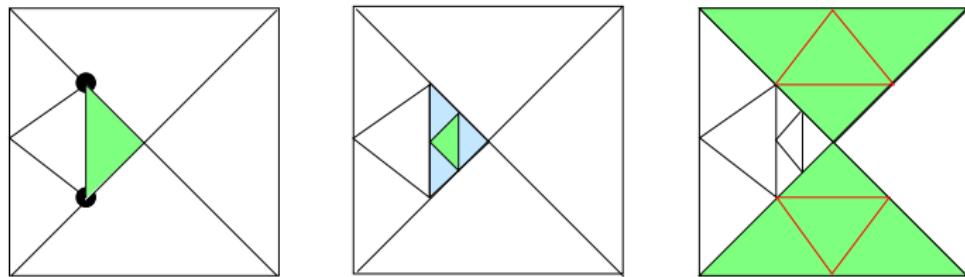
Context

Discontinuous Galerkin (DG) methods: some generalities

- Initially introduced to solve neutron transport problems
(W. Reed and T. Hill, 1973)
- Became popular as a framework for solving hyperbolic or mixed hyperbolic/parabolic problems
- Somewhere between a finite element and a finite volume method, gathering many good features of both
- Main properties
 - Can handle unstructured, non-conforming meshes
 - Can easily deal with discontinuous coefficients and solutions
 - Yield local finite element mass matrices
 - Naturally lead to discretization (h -) and interpolation order (p -) adaptivity
 - Can handle elements of various types and shapes
 - Amenable to efficient parallelization

Context

Non-conforming simplicial meshes



- Red (non-conforming) refinement
 - Each triangle is split into 4 similar triangles
 - Each tetrahedron is split into 8 non-similar tetrahedra
- Can be used for more flexibility in the discretisation of :
 - complex domains,
 - heterogeneous media.
- Expected to reduce memory consumption and computing time

Content

① DGTD- \mathbb{P}_{p_i} method

- Formulation
- Properties

② hp -like DGTD- $\mathbb{P}_{(p_1, p_2)}$ method

- Numerical dispersion

③ Numerical results: 2D case

- Numerical convergence
- Computational cost

④ Closure

DGTD- \mathbb{P}_{p_i} method

- Time-domain Maxwell's equations

$$\bar{\epsilon} \partial_t \vec{\mathbf{E}} - \operatorname{curl} \vec{\mathbf{H}} = 0 \quad \text{and} \quad \bar{\mu} \partial_t \vec{\mathbf{H}} + \operatorname{curl} \vec{\mathbf{E}} = 0$$

- Boundary conditions : $\partial\Omega = \Gamma_a \cup \Gamma_m$

$$\begin{cases} \mathbf{n} \times \vec{\mathbf{E}} = 0 & \text{on } \Gamma_m \\ \mathbf{n} \times \vec{\mathbf{E}} - \sqrt{\frac{\mu}{\epsilon}} \mathbf{n} \times (\vec{\mathbf{H}} \times \mathbf{n}) = \mathbf{n} \times \vec{\mathbf{E}}_{\text{inc}} - \sqrt{\frac{\mu}{\epsilon}} \mathbf{n} \times (\vec{\mathbf{H}}_{\text{inc}} \times \mathbf{n}) & \text{on } \Gamma_a \end{cases}$$

- Triangulation of Ω : $\overline{\Omega_h} \equiv \mathcal{T}_h = \bigcup \overline{\tau_i}$

- Hanging nodes are allowed
- $a_{ik} = \tau_i \cap \tau_k$ (interface)
- $p = \{p_i : \tau_i \in \mathcal{T}_h\}$, p_i is the local polynomial degree
- Approximation space: $V_p(\mathcal{T}_h) := \{\mathbf{v} \in L^2(\Omega)^3 : \mathbf{v}|_{\tau_i} \in \mathbb{P}_{p_i}(\tau_i), \forall \tau_i \in \mathcal{T}_h\}$

DGTD- \mathbb{P}_{p_i} method

Space discretization

- Variational formulation: $\forall \vec{\varphi} \in \text{Span}\{\vec{\varphi}_{ij}, 1 \leq j \leq d_i\}$

$$\left\{ \begin{array}{lcl} \int_{\tau_i} \vec{\varphi} \cdot \bar{\epsilon}_i \partial_t \vec{\mathbf{E}} & = & \int_{\tau_i} \text{curl} \vec{\varphi} \cdot \vec{\mathbf{H}} - \int_{\partial \tau_i} \vec{\varphi} \cdot (\vec{\mathbf{H}} \times \mathbf{n}) \\ \int_{\tau_i} \vec{\varphi} \cdot \bar{\mu}_i \partial_t \vec{\mathbf{H}} & = & - \int_{\tau_i} \text{curl} \vec{\varphi} \cdot \vec{\mathbf{E}} + \int_{\partial \tau_i} \vec{\varphi} \cdot (\vec{\mathbf{E}} \times \mathbf{n}) \end{array} \right.$$

- Centered fluxes [M. Remaki : 2000, COMPEL]

$$\vec{\mathbf{E}}|_{a_{ik}} = \frac{\vec{\mathbf{E}}_i + \vec{\mathbf{E}}_k}{2}, \quad \vec{\mathbf{H}}|_{a_{ik}} = \frac{\vec{\mathbf{H}}_i + \vec{\mathbf{H}}_k}{2} \quad (1)$$

- Replacing surface integrals using (1), and re-integrating by parts

$$\left\{ \begin{array}{lcl} \int_{\tau_i} \vec{\varphi} \cdot \bar{\epsilon}_i \partial_t \vec{\mathbf{E}}_i & = & \frac{1}{2} \int_{\tau_i} (\text{curl} \vec{\varphi} \cdot \vec{\mathbf{H}}_i + \text{curl} \vec{\mathbf{H}}_i \cdot \vec{\varphi}) - \frac{1}{2} \sum_{k \in \mathcal{V}_i} \int_{a_{ik}} \vec{\varphi} \cdot (\vec{\mathbf{H}}_k \times \vec{n}_{ik}) \\ \int_{\tau_i} \vec{\varphi} \cdot \bar{\mu}_i \partial_t \vec{\mathbf{H}}_i & = & - \frac{1}{2} \int_{\tau_i} (\text{curl} \vec{\varphi} \cdot \vec{\mathbf{E}}_i + \text{curl} \vec{\mathbf{E}}_i \cdot \vec{\varphi}) + \frac{1}{2} \sum_{k \in \mathcal{V}_i} \int_{a_{ik}} \vec{\varphi} \cdot (\vec{\mathbf{E}}_k \times \vec{n}_{ik}) \end{array} \right.$$

DGTD- \mathbb{P}_{p_i} method

- Matrix form of the DGTD- \mathbb{P}_{p_i} scheme :

$$\left\{ \begin{array}{lcl} M_i^\epsilon \partial_t \mathbf{E}_i & = & K_i \mathbf{H}_i - \sum_{k \in \mathcal{V}_i} S_{ik} \mathbf{H}_k \\ M_i^\mu \partial_t \mathbf{H}_i & = & -K_i \mathbf{E}_i + \sum_{k \in \mathcal{V}_i} S_{ik} \mathbf{E}_k \end{array} \right.$$

- M_i^ϵ and M_i^μ are the symmetric positive definite mass matrices of size d_i
- K_i is the symmetric stiffness matrix of size d_i
- S_{ik} is the interface matrix of size $d_i \times d_k$:

$$(S_{ik})_{jl} = \frac{1}{2} \int_{a_{ik}} \varphi_j \cdot (\psi_l \times \mathbf{n}_{ik})$$

- If a_{ik} is a conforming interface \Rightarrow no problem
- If a_{ik} is a non-conforming interface \Rightarrow we calculate S_{ik} using the Gauss-Legendre numerical quadrature [Fahs et al : 2007, RR-6162, INRIA]

DGTD- \mathbb{P}_{p_i} method

Time discretization

- H. Spachmann, R. Schuhmann and T. Weiland : Int. J. Numer. Model., 2002
- J.L. Young : Radio Science, 2001

- High-order Leap-frog (LF_N) time scheme

- Unknowns related to \mathbf{E} are approximated at $t^n = n\Delta t$
- Unknowns related to \mathbf{H} are approximated at $t^{n+\frac{1}{2}} = (n + \frac{1}{2})\Delta t$

$$\left\{ \begin{array}{ll} \mathbf{T}_1 = \Delta t (M_i^\epsilon)^{-1} \operatorname{curl}(\vec{\mathbf{H}}_i^{n+\frac{1}{2}}), & \mathbf{T}_1^* = -\Delta t (M_i^\mu)^{-1} \operatorname{curl}(\vec{\mathbf{E}}_i^{n+1}), \\ \mathbf{T}_2 = -\Delta t (M_i^\mu)^{-1} \operatorname{curl}(\mathbf{T}_1), & \mathbf{T}_2^* = \Delta t (M_i^\epsilon)^{-1} \operatorname{curl}(\mathbf{T}_1^*), \\ \mathbf{T}_3 = \Delta t (M_i^\epsilon)^{-1} \operatorname{curl}(\mathbf{T}_2), & \mathbf{T}_3^* = -\Delta t (M_i^\mu)^{-1} \operatorname{curl}(\mathbf{T}_2^*). \\ \\ \textcolor{blue}{LF}_2 : \left\{ \begin{array}{lcl} \mathbf{E}_i^{n+1} & = & \mathbf{E}_i^n + \mathbf{T}_1 \\ \mathbf{H}_i^{n+\frac{3}{2}} & = & \mathbf{H}_i^{n+\frac{1}{2}} + \mathbf{T}_1^* \end{array} \right. \\ \\ \textcolor{blue}{LF}_4 : \left\{ \begin{array}{lcl} \mathbf{E}_i^{n+1} & = & \mathbf{E}_i^n + \mathbf{T}_1 + \mathbf{T}_3/24 \\ \mathbf{H}_i^{n+\frac{3}{2}} & = & \mathbf{H}_i^{n+\frac{1}{2}} + \mathbf{T}_1^* + \mathbf{T}_3^*/24 \end{array} \right. \end{array} \right.$$

DGTD- \mathbb{P}_{p_i} method

General form

- General form of the DGTD- \mathbb{P}_{p_i} method :

$$\begin{cases} \mathbb{M}^\epsilon \frac{\mathbb{E}^{n+1} - \mathbb{E}^n}{\Delta t} = \mathbb{S}_N \mathbb{H}^{n+\frac{1}{2}}, \\ \mathbb{M}^\mu \frac{\mathbb{H}^{n+\frac{3}{2}} - \mathbb{H}^{n+\frac{1}{2}}}{\Delta t} = - {}^t \mathbb{S}_N \mathbb{E}^{n+1}, \end{cases}$$

where the $d \times d$ matrix \mathbb{S}_N verifies:

$$\mathbb{S}_N = \begin{cases} \mathbb{S} & \text{if } N = 2, \\ \mathbb{S}(\mathbb{I} - \frac{\Delta t^2}{24} \mathbb{M}^{-\mu} {}^t \mathbb{S} \mathbb{M}^{-\epsilon} \mathbb{S}) & \text{if } N = 4. \end{cases}$$

- \mathbb{E} and \mathbb{H} are of size $d = \sum_i d_i$
- \mathbb{M}^ϵ and \mathbb{M}^μ are block diagonal mass matrices of size d with diagonal blocks equal to M_i^ϵ and M_i^μ respectively

Properties of the DGTD- \mathbb{P}_{p_i} method

Stability

- Consider the following discrete electromagnetic energies:

- Local energy : $\mathcal{E}_i^n = \frac{1}{2} ({}^t \mathbf{E}_i^n M_i^\epsilon \mathbf{E}_i^n + {}^t \mathbf{H}_i^{n-\frac{1}{2}} M_i^\mu \mathbf{H}_i^{n+\frac{1}{2}})$

- Total energy : $\mathcal{E}^n = \frac{1}{2} ({}^t \mathbb{E}^n \mathbb{M}^\epsilon \mathbb{E}^n + {}^t \mathbb{H}^{n-\frac{1}{2}} \mathbb{M}^\mu \mathbb{H}^{n+\frac{1}{2}})$

- The total energy \mathcal{E}^n is exactly conserved (when $\Gamma_a = \emptyset$)
- The total energy \mathcal{E}^n is stable if

$$\Delta t \leq \frac{2}{d_N}, \text{ with } d_N = \|\mathbb{M}^{-\frac{\mu}{2}} {}^t \mathbb{S}_N \mathbb{M}^{-\frac{\epsilon}{2}}\|,$$

- CFL(LF₄) \simeq **2.847** \times CFL(LF₂)
- Elements of the Proof
 - Develop $\mathbb{H}^{n+\frac{1}{2}}$ in function of $\mathbb{H}^{n-\frac{1}{2}}$ and \mathbb{E}^n
 - Prove that \mathcal{E}^n is a positive definite quadratic form of all unknowns ($\mathbb{E}^n, \mathbb{H}^{n-\frac{1}{2}}$)

Properties of the DGTD- \mathbb{P}_{p_i} method

Stability and convergence

- The local discrete energy \mathcal{E}_i^n is stable if

$$\forall i, \forall k \in \mathcal{V}_i, \quad c_i \Delta t [2\alpha_i + \beta_{ik}] < \frac{4V_i}{P_i}.$$

- The dimensionless constants α_i and β_{ik} ($k \in \mathcal{V}_i$) verify

$$\forall \vec{\mathbf{X}} \in \mathcal{P}_i, \quad \begin{cases} \|\operatorname{curl} \vec{\mathbf{X}}\|_{\tau_i} \leq \frac{\alpha_i P_i}{V_i} \|\vec{\mathbf{X}}\|_{\tau_i}, \\ \|\vec{\mathbf{X}}\|_{a_{ik}}^2 \leq \frac{\beta_{ik} S_{ik}}{V_i} \|\vec{\mathbf{X}}\|_{\tau_i}^2, \end{cases}$$

- Numerical CFL values for the DGTD- \mathbb{P}_p method

p	0	1	2	3	4	5	6
CFL(LF ₂)	1.0	0.3	0.2	0.1	0.08	0.06	0.045

- Convergence analysis [Fezoui *et al* : 2005, M2AN]

$$\mathcal{O}(Th^{\min(s,p)}) + \mathcal{O}(\Delta t^N)$$

- The asymptotic convergence order is bounded by N independently of p

Properties of the DGTD- \mathbb{P}_{p_i} method

Numerical dispersion

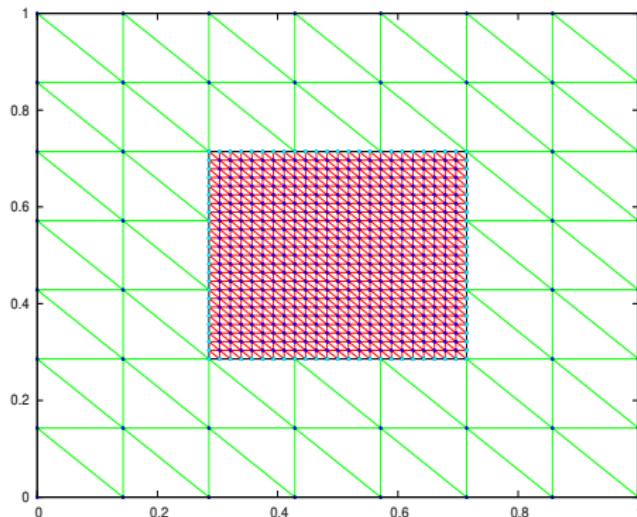
- Two-dimensional Maxwell's equation (TMz)

$$\begin{cases} \epsilon \frac{\partial E_z}{\partial t} - \frac{\partial H_y}{\partial x} + \frac{\partial H_x}{\partial y} = 0 \\ \mu \frac{\partial H_x}{\partial t} + \frac{\partial E_z}{\partial y} = 0 \\ \mu \frac{\partial H_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0 \end{cases}$$

- Eigenmode in a unitary PEC square cavity
- $f = 0.212$ GHz, $p_i = p = \text{constant}$
- Simulations are carried out for $t = 60$ (43 periods)
- 7-irregular non-conforming meshes (**a centered zone is refined 3 times**)
 - For $p = 0, 1 \Rightarrow 10$ points per wavelength
 - For $p = 2, 3, 4 \Rightarrow 6$ points per wavelength

Eigenmode in a PEC cavity

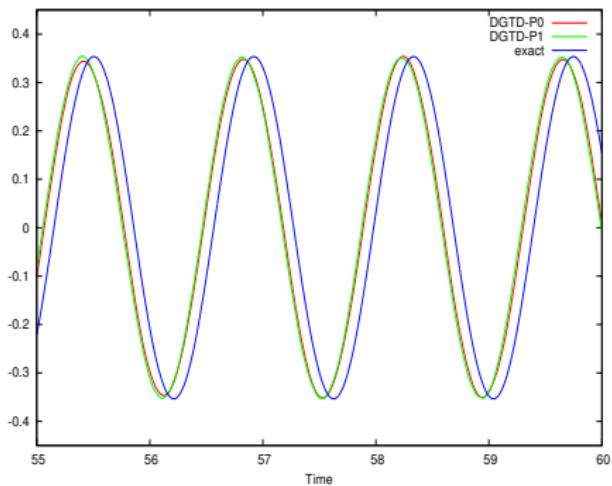
Non-conforming mesh



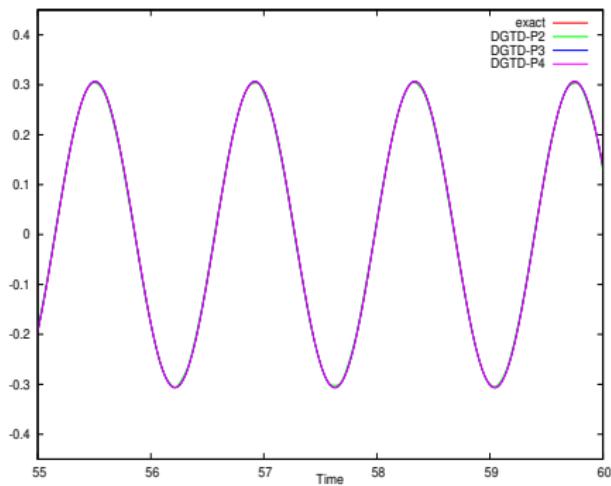
The centered zone is locally refined 3 times
7-irregular mesh

Numerical dispersion

DGTD- \mathbb{P}_p , $p \leq 1$ method



DGTD- \mathbb{P}_p , $p \geq 2$ method



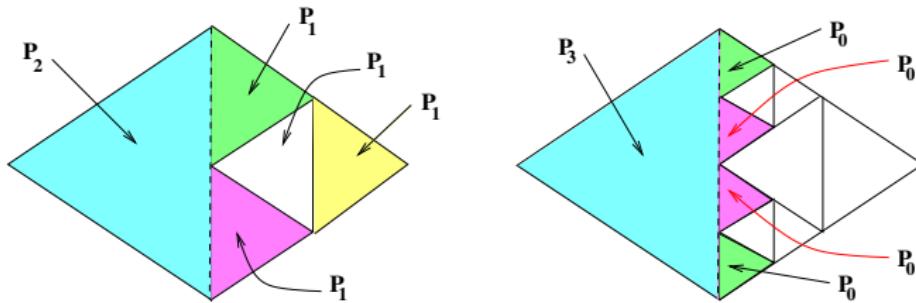
DGTD- \mathbb{P}_p method : time evolution of the H_x component
Zoom on the last 5 periods

DGTD- $\mathbb{P}_{(p_1, p_2)}$ method

→ H. Fahs, L. Fezoui, S. Lanteri and F. Rapetti : IEEE Trans. Magn., 2008

- The DGTD- $\mathbb{P}_{(p_1, p_2)}$ method consists in using:
 - high polynomial degrees " p_1 " in the coarse elements,
 - low polynomial degrees " p_2 " in the refined elements.
- If $p_1 = p_2 = p$, we find again the classical DGTD- \mathbb{P}_p method
- Numerical CFL values for the DGTD- $\mathbb{P}_{(p_1, p_2)}$ method

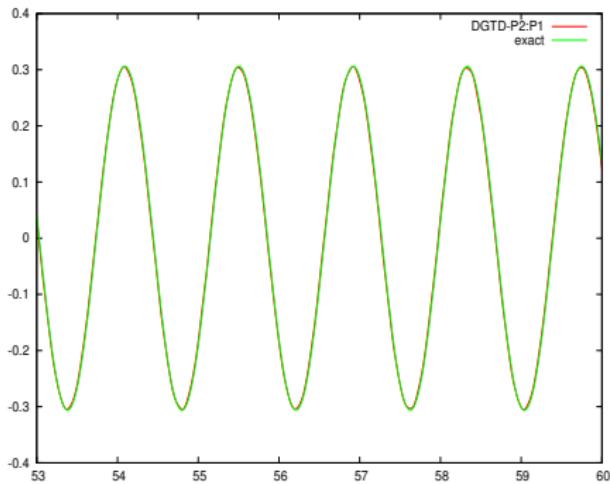
(p_1, p_2)	(2,1)	(3,2)	(4,2)	(4,3)	(5,3)	(5,4)
CFL(LF ₂)	0.3	0.2	0.2	0.1	0.1	0.08



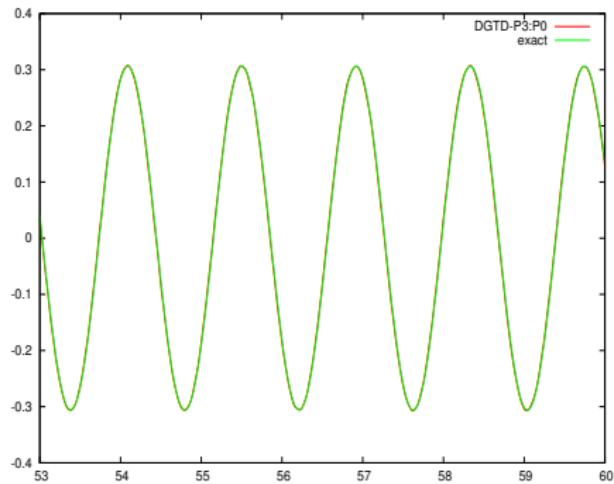
DGTD- $\mathbb{P}_{(p_1, p_2)}$ method

Eigenmode in a PEC cavity

DGTD- $\mathbb{P}_{(2,1)}$ method



DGTD- $\mathbb{P}_{(3,0)}$ method



DGTD- $\mathbb{P}_{(p_1, p_2)}$ method : time evolution of the H_x component
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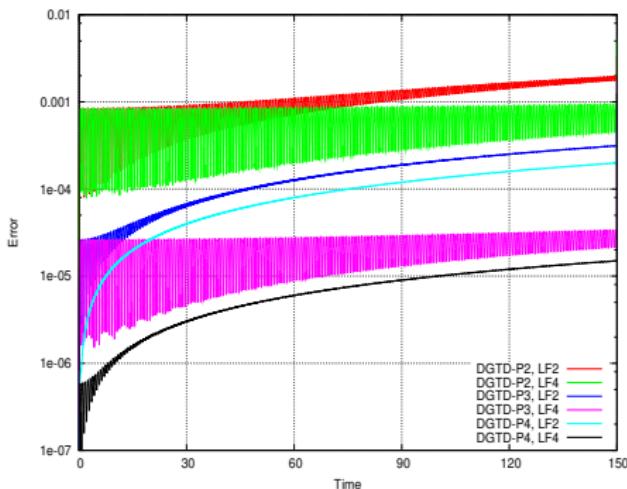
Numerical results

Eigenmode in a PEC cavity

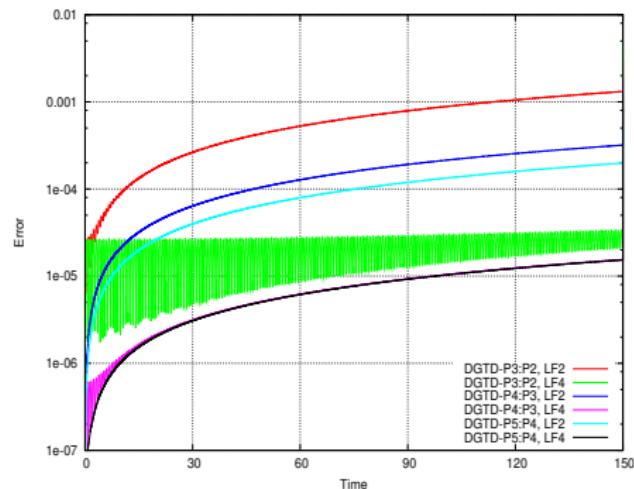
Comparison between LF2/LF4 and DGTD- \mathbb{P}_p /DGTD- $\mathbb{P}_{(p_1, p_2)}$ methods

Non-conforming mesh: 782 triangles and 442 nodes (36 hanging nodes)

LF2/LF4 DGTD- \mathbb{P}_p method



LF2/LF4 DGTD- $\mathbb{P}_{(p_1, p_2)}$ method



Time evolution of the L^2 error for $t = 150$ (106 periods)

Numerical results

Eigenmode in a PEC cavity

Comparison between LF_2/LF_4 and $\text{DGTD-}\mathbb{P}_p/\text{DGTD-}\mathbb{P}_{(p_1,p_2)}$ methods

Table: # DOF, L^2 -errors and CPU time using the LF_2 and LF_4 DGTD methods

DGTD- \mathbb{P}_p method		LF_2		LF_4	
p	# DOF	Error	CPU (min)	Error	CPU (min)
2	4692	1.8E-03	11	5.5E-04	8
3	7820	3.1E-04	39	2.4E-05	28
4	11730	1.9E-04	98	1.5E-05	70
5	16422	1.5E-04	220	1.3E-05	155

DGTD- $\mathbb{P}_{(p_1,p_2)}$ method		LF_2		LF_4	
(p_1, p_2)	# DOF	Error	CPU (min)	Error	CPU (min)
(3,2)	6668	1.3E-03	17	2.3E-05	12
(4,2)	9138	1.3E-03	27	1.5E-05	19
(4,3)	10290	3.2E-04	61	1.5E-05	44
(5,4)	14694	2.0E-04	134	1.4E-05	95

Numerical results

Eigenmode in a PEC cavity

Comparison between LF_2/LF_4 and $\text{DGTD-}\mathbb{P}_p/\text{DGTD-}\mathbb{P}_{(p_1,p_2)}$ methods

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Numerical results

Eigenmode in a PEC cavity

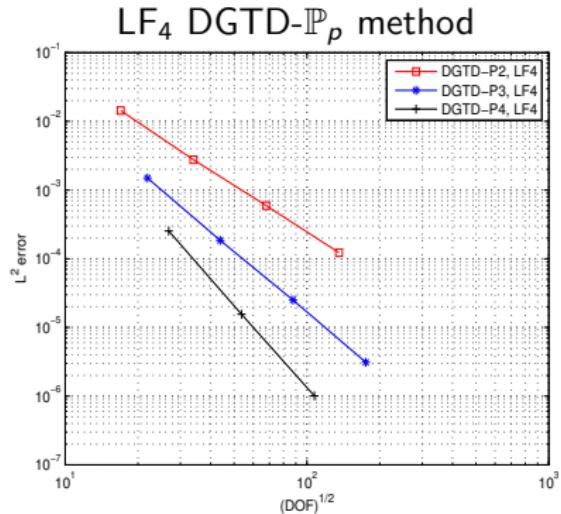
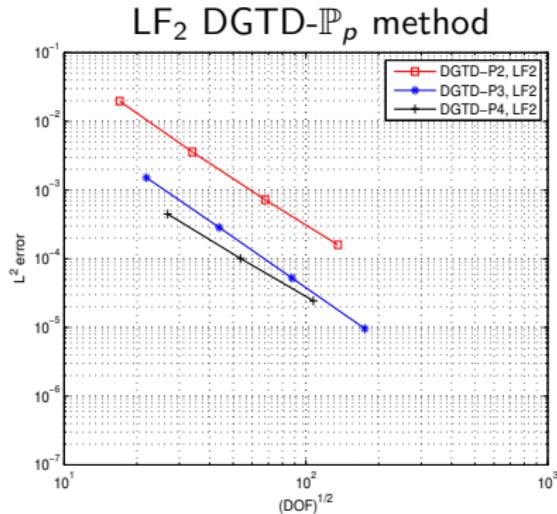


Table: Asymptotic convergence orders

$p =$	2	3	4
LF_2 scheme	2.28	2.33	2.10
LF_4 scheme	2.32	2.97	3.99

Numerical results

Eigenmode in a PEC cavity

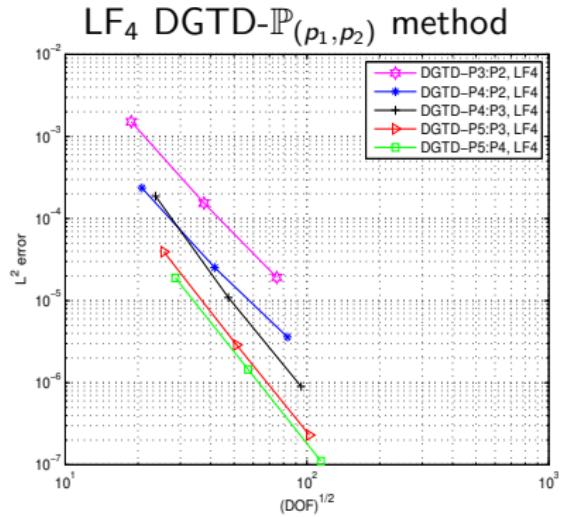
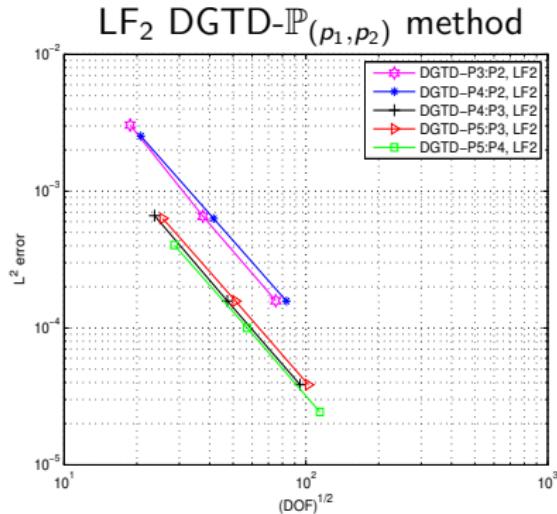


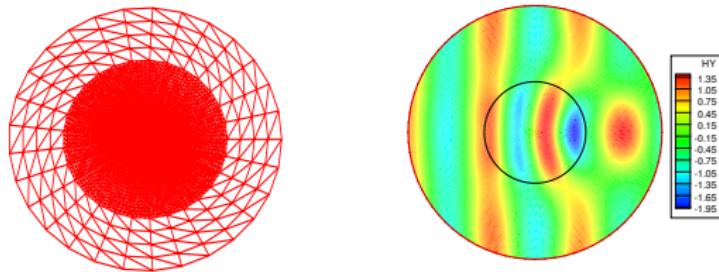
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$(p_1, p_2) =$	(3,2)	(4,2)	(4,3)	(5,3)	(5,4)
LF ₂ scheme	2.13	2.00	2.05	2.02	2.03
LF ₄ scheme	3.15	3.02	3.85	3.71	3.71

Numerical results

Scattering of a plane wave by a dielectric cylinder

- Conforming mesh: 11920 triangles and 6001 nodes
- Non-conforming mesh: 5950 triangles and 3151 nodes (300 hanging nodes)



DGTD- \mathbb{P}_p : Conforming triangular mesh

method	DGTD- \mathbb{P}_0	DGTD- \mathbb{P}_1	DGTD- \mathbb{P}_2	DGTD- \mathbb{P}_3
L^2 error, CPU (min)	13.6%, 20	7.15%, 178	5.20%, 542	5.22%, 1817
# DOF	11920	35760	71520	119200

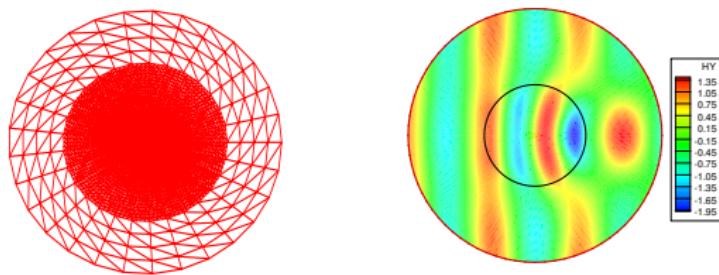
DGTD- $\mathbb{P}_{(p_1, p_2)}$: Non-conforming triangular mesh

method	DGTD- $\mathbb{P}_{(1,0)}$	DGTD- $\mathbb{P}_{(2,0)}$	DGTD- $\mathbb{P}_{(2,1)}$	DGTD- $\mathbb{P}_{(3,2)}$
L^2 error, CPU (min)	11.6%, 9	5.36%, 25	5.39%, 33	5.37%, 179
# DOF	11450	19700	26100	46700

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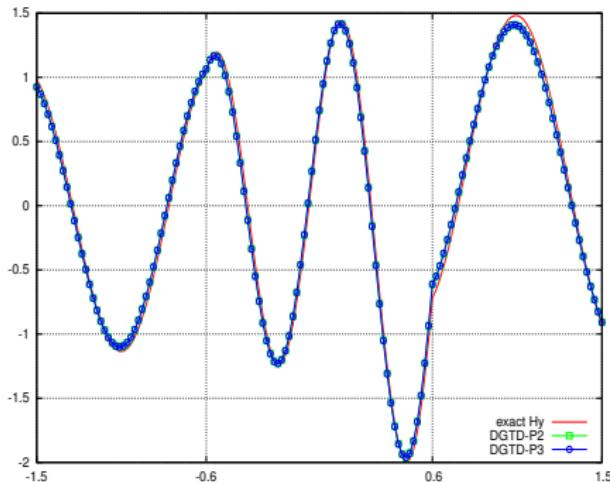
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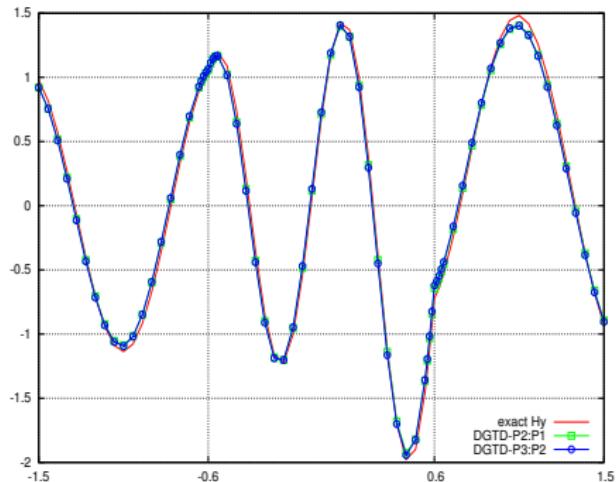
Numerical results

Scattering of a plane wave by a dielectric cylinder

DGTD- \mathbb{P}_2 & DGTD- \mathbb{P}_3 methods
Conforming mesh



DGTD- $\mathbb{P}_{(2,1)}$ & DGTD- $\mathbb{P}_{(3,2)}$ methods
Non-conforming mesh



1D distribution of H_y along $y = 0.0$ at $t = 5$

In progress

- Preliminary extension to 3D
- Design of an *a posteriori* error estimator for a *hp*-adaptive DGTD method

Thank you for your attention!

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